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EXPONENTS FOR B-STABLE IDEALS

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Abstract. Let G be a simple algebraic group over the complex numbers containing a Borel subgroup B. Given a B-stable ideal I in the nilradical of the Lie algebra of B, we define natural numbers $m_1, m_2, \ldots, m_k$ which we call ideal exponents. We then propose two conjectures where these exponents arise, proving these conjectures in types A_n, B_n, C_n and some other types.

When $I = 0$, we recover the usual exponents of G by Kostant (1959), and one of our conjectures reduces to a well-known factorization of the Poincaré polynomial of the Weyl group. The other conjecture reduces to a well-known result of Arnold-Brieskorn on the factorization of the characteristic polynomial of the corresponding Coxeter hyperplane arrangement.

1. Introduction

Let G be a simple algebraic group over the complex numbers containing a Borel subgroup B. The ideals in the nilradical of the Lie algebra of B, which are stable under the action of B, have recently attracted much attention.

In this paper we define a sequence of natural numbers $m_1, m_2, \ldots, m_k$ for each B-stable ideal I, and call them ideal exponents. The definition is a generalization of the usual exponents of G in the case where $I = 0$, via Kostant's proof relating the heights of positive roots to the exponents [Ko1].

We then conjecture (and prove in type A_n, B_n, C_n and in some other cases) two results about these ideal exponents. The first concerns a Poincaré polynomial defined for each ideal which generalizes the Poincaré polynomial for the Weyl group. The conjecture is that this new polynomial factors according to the ideal exponents just as the usual polynomial factors according to the usual exponents. This result is relevant for the study of regular nilpotent Hessenberg varieties (there is one for each ideal) since the combinatorially defined Poincaré polynomials in this paper should be the actual Poincaré polynomials for these varieties. This is known in many cases, as studied in [Ty].

The second occurrence of these new exponents is in the context of a hyperplane arrangement defined for each ideal. The hyperplane arrangement in question consists of those hyperplanes which correspond to the positive roots whose root space does not belong to the ideal. Generalizing the known result that the usual exponents are the roots of the characteristic polynomial for the full Coxeter arrangement, we conjecture (and prove in the classical types) that the characteristic polynomial of this new hyperplane arrangement has (non-trivial) roots $m_1, m_2, \ldots, m_k$. We also

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speculate that these arrangements are free (which we also prove in the classical types).

The paper concludes with speculation linking these two occurrences of the ideal exponents.

2. Notation

Fix a maximal torus T in B and let $(X, \Phi, Y, \Phi^\vee)$ be the root datum determined by G and T, and let W be the Weyl group. Let $\Pi \subset \Phi^+$ denote the simple roots and positive roots determined by B. As usual, $\langle\ ,\ \rangle$ denotes the pairing of X and Y. Let $Q^\vee$ denote the lattice in Y generated by $\Phi^\vee$ (the coroot lattice). We denote the standard partial order on Φ by $\preceq$; so $\alpha \preceq \beta$ for $\alpha, \beta \in \Phi$ if and only if $\beta - \alpha$ is a sum of positive roots. As is customary, we write $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$. For $\beta \in \Phi$, write $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ and let $\operatorname{ht}(\beta) = \sum_{\alpha\in\Pi} c_\alpha$ denote the height of β. Given $\alpha \in \Phi$, let $s_\alpha \in W$ denote the corresponding reflection.

We define an ideal (also called an upper order ideal) $\mathcal{I}$ of Φ^+ to be a collection of roots such that if $\alpha \in \mathcal{I}, \beta \in \Phi^+$, and $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in \mathcal{I}$. In other words, if $\alpha \in \mathcal{I}$ and $\gamma \in \Phi^+$ with $\alpha \preceq \gamma$, then $\gamma \in \mathcal{I}$.

Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ be the Lie algebras of G, B, T, respectively. It is easy to see that B-stable ideals in the nilradical $\mathfrak{n}$ of $\mathfrak{b}$ are naturally in bijection with the ideals of Φ^+. Namely, if I is a B-stable ideal of $\mathfrak{n}$, it is stable under the action of T, hence I is a sum of root spaces. Denote by $\mathcal{I}$ the set of roots whose root space is contained in I. Then $\mathcal{I}$ is an ideal of Φ^+ and this map is a bijection.

3. Ideal exponents

In this section, motivated by Kostant's proof relating the heights of the positive roots and the usual exponents of G, we define exponents for each ideal. Our definition is an easy modification: we consider only those positive roots which do not lie in the ideal.

For an ideal $\mathcal{I} \subset \Phi^+$, let $\mathcal{I}^c = \Phi^+ - \mathcal{I}$ be the positive roots not in $\mathcal{I}$. Define

$$\lambda_i = \#\{\alpha \in \mathcal{I}^c \mid \operatorname{ht}(\alpha) = i\}.$$

We first observe

Proposition 3.1. *The λ_i give a partition of the number of roots in $\mathcal{I}^c$. That is,*

$$\lambda_1 \geq \lambda_2 \geq \dots .$$

In addition, $\lambda_1 > \lambda_2$.

Proof. This is easy to check in the classical groups and was checked on a computer in the exceptional groups. □

Let $k = \lambda_1$, which is just the number of simple roots in $\mathcal{I}^c$. We define $m_k^{\mathcal{I}} \geq \cdots \geq m_1^{\mathcal{I}}$ to be the dual partition of λ_i. In other words, $m_i^{\mathcal{I}} = \#\{\lambda_j \mid \lambda_j \geq k - i + 1\}$.

Definition 3.2. The **ideal exponents** of $\mathcal{I}$, also called $\mathcal{I}$-exponents, are the natural numbers

$$m_k^{\mathcal{I}} \geq m_{k-1}^{\mathcal{I}} \geq \cdots \geq m_1^{\mathcal{I}}.$$

It follows from the fact that $\lambda_1 > \lambda_2$ that $m_1^{\mathcal{I}} = 1$. We also observe, as mentioned previously, that when $\mathcal{I} = \emptyset$ these are the usual exponents (in this case k equals the rank of G) [Ko1].

We suspect that there are many situations where these new exponents will arise. We propose two situations in what follows, namely Theorem 4.1 and Theorem 11.1.

4. Poincaré polynomials for ideals

Let $R \subset \Phi^+$ be any subset of the positive roots. Given $S \subset R$ we say that S is **R-closed** if $\alpha, \beta \in S$ and $\alpha + \beta \in R$, and then also $\alpha + \beta \in S$.

Given an ideal $\mathcal{I} \subset \Phi^+$, we are interested in those subsets S of $\mathcal{I}^c$ with the property that both S and its complement $S^c := \mathcal{I}^c - S$ are $\mathcal{I}^c$-closed.

These subsets are analogous to Weyl group elements. Indeed, if $\mathcal{I} = \emptyset$ and if $w \in W$, then

$$N(w) := \{\alpha \in \Phi^+ \mid w(\alpha) \prec 0\},$$

and its complement in Φ^+ are both $\mathcal{I}^c$-closed (in this case, $\mathcal{I}^c = \Phi^+$). Conversely every subset of Φ^+ which is Φ^+-closed and whose complement in Φ^+ is Φ^+-closed is equal to $N(w)$ for a unique $w \in W$. This is well known and goes back to [Ko2].

Given this background, we define a subset S of $\mathcal{I}^c$ to be of **Weyl type** for $\mathcal{I}$ if both S and S^c are $\mathcal{I}^c$-closed. Let $\mathcal{W}^{\mathcal{I}}$ denote the subsets of $\mathcal{I}^c$ of Weyl type. One of the main results of this paper can now be formulated.

Theorem 4.1. *Let $\mathcal{I}$ be an ideal in Φ^+. Then in types $A_n, B_n, C_n, G_2, F_4, E_6$*

$$\sum_{S \in \mathcal{W}^{\mathcal{I}}} t^{|S|} = \prod_{i=1}^{k} (1 + t + t^2 + \cdots + t^{m_i^{\mathcal{I}}}), \tag{1}$$

where the $m_i^{\mathcal{I}}$ are the exponents of $\mathcal{I}$.

We conjecture that the theorem also holds in the remaining cases. We defer the proof, which is case-by-case, until Section 8.

In the case where $\mathcal{I}^c = \Phi^+$, the theorem is well known [Ko1], [Ma]. On the one hand, the $\mathcal{I}$-exponents become the usual exponents as mentioned previously. On the other hand, if $l(w)$ denotes the length of $w \in W$ with respect to the set of simple reflections coming from Π, then $l(w) = |N(w)|$, and so the left-hand side of (1) is equal to

$$\sum_{w \in W} t^{l(w)}.$$

Then (1) becomes the well-known factorization of the Poincaré polynomial of the Weyl group, which is also the Poincaré polynomial of the flag variety G/B when t is replaced by t^2.

5. Some old results for general root systems

Many of the results in this paper rely on the following lemma and its corollary. The lemma is well known. We thank Jim Humphreys for simplifying our earlier proof. Jantzen has pointed out to us that Joseph proves some of the equivalences of the lemma in [Jo].

Lemma 5.1. *Let $x, y \in W$. Then the following four conditions are equivalent:*

(i) $N(x) \subseteq N(yx)$.
(ii) $x^{-1}N(y) \subseteq \Phi^+$.

(iii) $N(yx) = N(x) \cup x^{-1}N(y)$.
(iv) $l(yx) = l(y) + l(x)$.

Proof. Certainly (iii) implies (ii) by definition. Next

$$N(x) \subseteq N(yx) \text{ if and only if } x^{-1}N(y) \subseteq \Phi^+ \tag{2}$$

since $\alpha \in x^{-1}N(y)$ with $\alpha \prec 0$ if and only if $-\alpha \in N(x)$ and $-\alpha \notin N(yx)$. This shows the equivalence of (i) and (ii).

It is easy to check from the definitions that

$$N(yx) \subseteq N(x) \cup x^{-1}N(y), \tag{3}$$

where the union is a disjoint union. Also

$$l(yx) \leq l(x) + l(y), \tag{4}$$

which implies that

$$|N(yx)| \leq |N(x)| + |N(y)| = |N(x)| + |x^{-1}N(y)|.$$

Hence equality in (4) is equivalent to equality in (3), which shows that (iii) and (iv) are equivalent. Finally, it is clear that

$$x^{-1}N(y) \cap \Phi^+ \subseteq N(yx),$$

so if (i) holds so does (iii) by (2) and (3). □

The next corollary follows from the lemma by taking a reduced expression for $y = wx^{-1} = s_\beta \cdots s_\alpha$, where $\alpha, \beta \in \Pi$.

Corollary 5.2. *Given $x, w \in W$. If $N(x) \subsetneq N(w)$, then there exists $\alpha, \beta \in \Pi$ so that*

$$N(x) \subsetneq N(s_\alpha x) \subseteq N(w) \quad \text{and}$$
$$N(x) \subseteq N(s_\beta w) \subsetneq N(w).$$

We conclude the section with a well-known lemma related to the minimal length coset representatives of a parabolic subgroup of W. We include a proof since we refer to the proof in what follows.

Let $\Phi' \subset \Phi$ be a parabolic subsystem. In other words, Φ' has a basis of simple roots which is contained in the simple roots Π of Φ. Let $\Phi^1 = \Phi^+ - \Phi'^+$ and let $W' \subset W$ be the Weyl group of Φ'.

Lemma 5.3. *The set $C := \{x \in W \mid N(x) \subseteq \Phi^1\}$ is a set of distinct coset representatives for W' in W.*

Proof. Take $w \in W$. The intersection $N(w) \cap \Phi'$ is of Weyl type for Φ'. Thus

$$N(w) \cap \Phi' = N(x)$$

for some $x \in W'$, where $N(x)$ is the same whether computed in W' or W. Now $N(x) \subseteq N(w)$, so by Lemma 5.1, we have that $y = wx^{-1}$ satisfies

$$x^{-1}N(y) = N(w) - N(x).$$

Hence $x^{-1}N(y) \subseteq \Phi^1$ by the definition of x. Consequently $N(y) \subseteq \Phi^1$ as $x \in W'$ and W' preserves Φ^1. Certainly, $wW' = yxW' = yW'$, which shows that the elements of C are a set of coset representatives.

They must be distinct representatives. Indeed, suppose $y = zx$ for $y, z \in C$ and some $x \in W'$. Then $x^{-1}N(z) \subseteq \Phi^1$ since W' preserves Φ^1. Then Lemma 5.1 implies that $N(x) \subseteq N(y) \subseteq \Phi^1$, forcing $x = 1$ since also $N(x) \subseteq \Phi'$ and so $y = z$. □

6. Some new results for general root systems

The first new result of this paper is a generalization of the fact that every subset of Φ^+ of Weyl type is of the form $N(w)$ for some $w \in W$ [Ko2]. On the one hand, it is easy to see that $N(w) \cap \mathcal{I}^c$ for $w \in W$ is always of Weyl type in $\mathcal{I}^c$. But the converse is also true. Namely, every subset S of $\mathcal{I}^c$ of Weyl type is of the form $N(w) \cap \mathcal{I}^c$ for some $w \in W$. That is,

Proposition 6.1. *Let $S \in \mathcal{W}^{\mathcal{I}}$. There exists $w \in W$ such that $S = N(w) \cap \mathcal{I}^c$.*

Proof. This is equivalent to showing that there exists $T \in \mathcal{W}^{\emptyset}$ for which $S = T \cap \mathcal{I}^c$, since we already know the result is true when $\mathcal{I} = \emptyset$. In fact, it is enough to prove that given an ideal $\mathcal{I}_1$ where $\mathcal{I}_1^c = \mathcal{I}^c \cup \{\delta\}$ for $\delta \in \Phi^+$, we can find $T \in \mathcal{W}^{\mathcal{I}_1}$ where $S = T \cap \mathcal{I}^c$. Then the result would follow by induction as there is always a sequence

$$\mathcal{I}^c = \mathcal{I}_0^c \subset \mathcal{I}_1^c \subset \cdots \subset \mathcal{I}_r^c = \Phi^+,$$

such that $\mathcal{I}_i$ is an ideal and $|\mathcal{I}_{i+1}^c| = |\mathcal{I}_i^c| + 1$.

There are four possible situations given the above setup:

(1) $S,\ S \cup \{\delta\} \in \mathcal{W}^{\mathcal{I}_1}$.
(2) $S \in \mathcal{W}^{\mathcal{I}_1},\ S \cup \{\delta\} \notin \mathcal{W}^{\mathcal{I}_1}$.
(3) $S \notin \mathcal{W}^{\mathcal{I}_1},\ S \cup \{\delta\} \in \mathcal{W}^{\mathcal{I}_1}$.
(4) $S,\ S \cup \{\delta\} \notin \mathcal{W}^{\mathcal{I}_1}$.

Assuming one of the first three possibilities arises, one of S or $S \cup \{\delta\}$ (or in the first case both) would suffice for T. The proposition is therefore equivalent to the last possibility never occurring.

The last possibility would only occur if there exists $\alpha, \beta \in S$ and $\alpha', \beta' \in \mathcal{I}^c - S$ for which $\alpha + \beta = \delta = \alpha' + \beta'$. If so, then

$$-\alpha' + \alpha + \beta = \beta'.$$

By Lemma 3.2 in [So], either $-\alpha' + \alpha$ or $-\alpha' + \beta$ lies in $\Phi \cup \{0\}$. Since $\alpha, \beta \in S$ and $\alpha' \notin S$, neither $-\alpha' + \alpha$ nor $-\alpha' + \beta$ can be zero. Without loss of generality we take $-\alpha' + \alpha \in \Phi$, and by possibly interchanging the roles of α and α' in what follows, we may assume $-\alpha' + \alpha \in \Phi^+$.

On the one hand, $(-\alpha' + \alpha) + \alpha' = \alpha$. Now since $\mathcal{I}$ is an ideal and $\alpha \in \mathcal{I}^c$ and $-\alpha' + \alpha \prec \alpha$, then $-\alpha' + \alpha \in \mathcal{I}^c$. It follows that $-\alpha' + \alpha \in S$, since $\alpha' \notin S$ and $\alpha \in S$ and otherwise $S^c = \mathcal{I}^c - S$ would not be $\mathcal{I}^c$-closed.

On the other hand, $-\alpha' + \alpha = \beta' - \beta \in \mathcal{I}^c$. Clearly, $(\beta' - \beta) + \beta = \beta'$. Then $\beta' - \beta \notin S$, since $\beta \in S$ and $\beta' \notin S$ and otherwise S would not be $\mathcal{I}^c$-closed. This contradicts the fact that $-\alpha' + \alpha = \beta' - \beta \in S$ from the previous paragraph.

It follows that T can be chosen to be one of S or $S \cup \{\delta\}$ (or possibly both), proving the proposition. □

Although there is not always a unique $w \in W$ satisfying the hypotheses of the proposition, there is a unique w with the property that $N(w)$ is contained in $N(w')$ for any other $w' \in W$ satisfying the hypotheses of the proposition. More generally,

Proposition 6.2. *Let $\mathcal{I}' \subseteq \mathcal{I}$ be ideals. Given $S \in \mathcal{W}^{\mathcal{I}}$, there exists $T \in \mathcal{W}^{\mathcal{I}'}$ with the property that $S = T \cap \mathcal{I}^c$, and if $\widehat{T} \in \mathcal{W}^{\mathcal{I}'}$ satisfies $S \subseteq \widehat{T}$, then $T \subseteq \widehat{T}$. Such a T is clearly unique.*

Proof. The proof is by induction on the difference in cardinalities $l = |\mathcal{I}| - |\mathcal{I}'|$. The case $l = 0$ is trivial, with $T = S$. If $l > 0$, pick an ideal $\mathcal{I}_1$ such that

$$\mathcal{I}' \subsetneq \mathcal{I}_1 \subseteq \mathcal{I},$$

where $\mathcal{I}'^c = \mathcal{I}_1^c \cup \{\delta\}$ for some $\delta \in \Phi^+$. By induction there exists $T_1 \in \mathcal{W}^{\mathcal{I}_1}$ satisfying the hypotheses of the proposition with respect to $\mathcal{I}_1 \subseteq \mathcal{I}$ and $S \in \mathcal{W}^{\mathcal{I}}$.

Set $T = T_1$ if $T_1 \in \mathcal{W}^{\mathcal{I}'}$, or else set $T = T_1 \cup \{\delta\}$ if $T_1 \notin \mathcal{W}^{\mathcal{I}'}$. In either case the proof of the previous proposition ensures that $T \in \mathcal{W}^{\mathcal{I}'}$.

Now suppose that $\widehat{T} \in \mathcal{W}^{\mathcal{I}'}$ satisfies $S \subseteq \widehat{T}$. Since $\mathcal{I}_1^c \subset \mathcal{I}'^c$, it is clear that $\widehat{T} \cap \mathcal{I}_1^c \in \mathcal{W}^{\mathcal{I}_1}$. Then the minimal property for T_1 gives that

$$T_1 \subseteq \widehat{T} \cap \mathcal{I}_1^c.$$

Hence $T_1 \subseteq \widehat{T}$. We deduce that $T \subseteq \widehat{T}$. Indeed, $T = T_1 \cup \{\delta\}$ only when there exist $\alpha, \beta \in T_1$ with $\alpha + \beta = \delta$. Since $T_1 \subseteq \widehat{T}$ and $\widehat{T}$ is $\mathcal{I}'^c$-closed, we must have $\delta \in \widehat{T}$. □

There is a nice characterization of the $w \in W$ for which $T = N(w)$ satisfies the minimal condition of Proposition 6.2 when $\mathcal{I}' = \emptyset$.

Proposition 6.3. *Given $S \in \mathcal{W}^{\mathcal{I}}$ there is a unique $w \in W$ satisfying both $S = N(w) \cap \mathcal{I}^c$ and*

$$w^{-1}(-\Pi) \cap \Phi^+ \subseteq \mathcal{I}^c. \tag{5}$$

Furthermore, $N(w) \subseteq N(y)$ for any $y \in W$ with $S \subseteq N(y)$.

Proof. Let w be such that $T = N(w)$ satisfies the minimal property from Proposition 6.2 for $\mathcal{I}' = \emptyset$ and S.

Suppose there exists a simple root $\alpha \in \Pi$ for which

$$w^{-1}(-\alpha) \in \Phi^+ - \mathcal{I}^c.$$

Consider $x = s_\alpha w$. Then

$$w^{-1}(-\alpha) = x^{-1}(\alpha) \in \Phi^+,$$

and so Lemma 5.1 implies that

$$N(w) = N(x) \cup \{w^{-1}(-\alpha)\}.$$

But $w^{-1}(-\alpha) \notin \mathcal{I}^c$ and therefore

$$N(x) \cap \mathcal{I}^c = N(w) \cap \mathcal{I}^c,$$

contradicting the minimal property of $N(w)$. Hence we must have

$$w^{-1}(-\Pi) \cap \Phi^+ \subseteq \mathcal{I}^c.$$

For the uniqueness, take $y \in W$ with $S = N(y) \cap \mathcal{I}^c$ and $y \neq w$. Then $N(w) \subsetneq N(y)$ by Proposition 6.2. By Corollary 5.2 there exists $\alpha \in \Pi$ such that

$$N(w) \subseteq N(s_\alpha y) \subsetneq N(y),$$

and this implies that $N(y) = N(s_\alpha y) \cup \{y^{-1}(-\alpha)\}$. It follows that $y^{-1}(-\alpha) \notin \mathcal{I}^c$ since

$$N(y) \cap \mathcal{I}^c = N(w) \cap \mathcal{I}^c$$

and hence $y^{-1}(-\Pi) \cap \Phi^+ \not\subseteq \mathcal{I}^c$. This shows that w is unique. □

7. Further results for types A_n, B_n, C_n

In this section we explore some properties which are particular to types A_n, B_n, C_n and which are used in the proof of Theorem 4.1.

We label the simple roots in types B_n and C_n so that α_n is the only simple root of its length. For each root system of type X_n, we embed X_{n-1} in X_n via the simple roots $\alpha_2, \ldots, \alpha_n$. Denote by Φ_{n-1} the roots of X_{n-1} and let

$$\Phi^1 = \Phi^+ - \Phi^+_{n-1}.$$

Let $W' \subset W$ be the Weyl group of X_{n-1}. One of the key facts about these root systems is that Φ^1 is linearly ordered under $\prec$ and there is only root of each height from 1 to the largest height (n in type A_n and $2n-1$ in types B_n, C_n).

In addition,

Lemma 7.1. *Given $\alpha, \beta \in \Phi^1$ with $\alpha \prec \beta$, we always have*

$$\beta - b\alpha = c\gamma \tag{6}$$

for some $\gamma \in \Phi_{n-1}$ and some $b, c \in \{1, 2\}$.

Proof. The roots in Φ^1 (in a standard basis) are

$$\begin{array}{ll} \{e_1 - e_j \mid j = 2, 3, \ldots, n\} & \text{for } A_{n-1}, \\ \{e_1 \pm e_j \mid j = 2, 3, \ldots, n\} \cup \{e_1\} & \text{for } B_n, \\ \{e_1 \pm e_j \mid j = 2, 3, \ldots, n\} \cup \{2e_1\} & \text{for } C_n. \end{array}$$

These roots are ordered as follows:

$$e_1 - e_2 \prec \cdots \prec e_1 - e_n \prec e_1 \prec e_1 + e_n \prec \cdots \prec e_1 + e_2 \prec 2e_1,$$

whenever the given root is present in the appropriate root system.

It is easy to see in type A_{n-1} that $\beta - \alpha \in \Phi^+_{n-2}$ (as desired, given the shift in subscript).

In type B_n, $\beta - \alpha \in \Phi^+_{n-1}$, unless $\beta = e_1 + e_j$ and $\alpha = e_1 - e_j$, in which case $\beta - \alpha = 2\gamma$ for some $\gamma \in \Phi^+_{n-1}$.

In type C_n, we have $\beta - \alpha \in \Phi^+_{n-1}$, except when $\beta = 2e_1$, the highest root. In that case, we can say that $\beta - 2\alpha \in \Phi_{n-1}$. □

Lemma 7.2. *In types A_n, B_n, C_n, for each $k \in \{0, 1, \ldots, |\Phi^1|\}$, there is a unique element $x \in W$ satisfying $N(x) \subseteq \Phi^1$ and $|N(x)| = k$.*

Proof. Certainly there exists a unique w^1 with $N(w^1) = \Phi^1$ since $\Phi^1 \in \mathcal{W}^\emptyset$. Explicitly, w^1 is the product of the long element of W and the long element of W', using, for example, Lemma 5.1. It follows that there exists at least one $x \in W$ satisfying $N(x) \subseteq \Phi^1$ and $|N(x)| = k$ from Corollary 5.2 by taking any reduced expression for w^1.

We still need to show uniqueness. Let $x \in W$ satisfy $N(x) \subsetneq \Phi^1$ and assume uniqueness is true when $k > |N(x)|$. Since $N(x) \subsetneq N(w^1)$, Corollary 5.2 implies the existence of $\alpha \in \Pi$ such that

$$N(x) \subsetneq N(s_\alpha x) \subseteq N(w^1),$$

where $N(s_\alpha x) = N(x) \cup \{x^{-1}(\alpha)\}$. Since $N(s_\alpha x) \subseteq N(w^1) = \Phi^1$, we have

$$x^{-1}(\alpha) \in \Phi^1.$$

We claim that α is unique. Indeed, assume that $\beta \in \Pi$ and $x^{-1}(\beta) \in \Phi^1$ with $\beta \neq \alpha$. Without loss of generality (since Φ^1 is linearly ordered), $x^{-1}(\alpha) \prec x^{-1}(\beta)$. By Lemma 7.1, $x^{-1}(\beta) - bx^{-1}(\alpha) = c\gamma$ for some b, c and $\gamma \in \Phi_{n-1}$. Applying x to both sides yields $\beta - b\alpha = cx(\gamma)$. This is impossible since the right side is a linear combination of simple roots with either all positive or all negative coefficients, whereas the left side is a combination of two simple roots whose coefficients have opposite signs. Since $s_\alpha x$ is unique by induction, x is unique and the result follows. □

8. Proof of Theorem 4.1 for types A_n, B_n, C_n

Assume the factorization is true for Φ_{n-1}. Clearly $\mathcal{I}^c \cap \Phi_{n-1}$ is equal to $\mathcal{I}'^c$ for some ideal $\mathcal{I}'$ for Φ_{n-1}. Let $m_1, \dots, m_{k-1}$ be the $\mathcal{I}'$-exponents.

As noted in the previous section, the roots of $\Phi^1 = \Phi^+ - \Phi^+_{n-1}$ are linearly ordered and so contain one root of each height. It follows that $\mathcal{I}^c \cap \Phi^1$ contains one root of height $1, 2, \dots, m_k$ for some natural number m_k, and that the $\mathcal{I}$-exponents are m_k together with the $\mathcal{I}'$-exponents $m_1, m_2, \dots, m_{k-1}$ (in this indexing, m_k need not be the largest exponent).

Now it is certainly true that if $S \in \mathcal{W}^{\mathcal{I}}$, then $S \cap \Phi_{n-1} \in \mathcal{W}^{\mathcal{I}'}$. We have a strong converse which holds in types A_n, B_n, C_n:

Lemma 8.1. *Let $S' \in \mathcal{W}^{\mathcal{I}'}$. For each $j \in \{0, 1, \dots, |\Phi^1|\}$, there exists a unique $S \in \mathcal{W}^{\mathcal{I}}$ with $S \cap \Phi_{n-1} = S'$ and $|S \cap \Phi^1| = j$.*

Proof. Let $w' \in W'$ be the unique element with the property that $N(w') \cap \mathcal{I}'^c = S'$ and $N(w') \subseteq N(x)$ for any $x \in W$ with $N(x) \cap \mathcal{I}'^c = S'$ as in Proposition 6.2.

Let $\{x_0 = 1, x_1, \dots, x_i, \dots\}$ be the elements from Lemma 7.2 with $|N(x_i)| = i$. Note that

$$N(x_i) \subsetneq N(x_{i+1})$$

from the existence part of the proof of that lemma.

Consider the elements $x_i w'$. As in the proof of Lemma 5.3,

$$N(x_i w') = N(w') \cup w'^{-1} N(x_i), \tag{7}$$

where $w'^{-1} N(x_i) \subseteq \Phi^1$. It follows that $N(x_i w') \subsetneq N(x_{i+1} w')$ and the two sets differ by a single element of Φ^1.

Next, consider the intersection

$$N(x_i w') \cap \mathcal{I}^c \cap \Phi^1.$$

This intersection is empty for $i = 0$ and has m_k elements when $i = n$ in type A_n and when $i = 2n - 1$ in types B_n and C_n. From the previous paragraph, we know that $N(x_{i+1} w') \cap \mathcal{I}^c \cap \Phi^1$ and $N(x_i w') \cap \mathcal{I}^c \cap \Phi^1$ can differ by at most one element. Consequently, for some i we have that $S := N(x_i w') \cap \mathcal{I}^c$ satisfies $S \cap \Phi_{n-1} = S'$ and $|S \cap \Phi^1| = j$ has the desired cardinality. This gives existence. It would also give uniqueness if we knew that every S is of the form $N(x_i w') \cap \mathcal{I}^c$ for some i.

To that end, suppose that $S \in \mathcal{W}^{\mathcal{I}}$ and $S \cap \Phi_{n-1} = S'$. Let $w \in W$ be the unique element with the property that $N(w) \cap \mathcal{I}^c = S$ and $N(w) \subseteq N(x)$ for any $x \in W$ with $S \subseteq N(x)$ as in Proposition 6.2.

Write $w = x_i y$ for $y \in W'$ by Lemma 5.3. By (7) and the line following it, $N(y) = N(w) \cap \Phi_{n-1}$. The latter contains S' by the definition of w. Thus there exists i' so that

$$S \subseteq N(x_{i'} w'),$$

simply by taking the largest possible value of i'. By Proposition 6.2, we get $N(x_i y) \subseteq N(x_{i'} w')$ and thus $N(y) \subseteq N(w')$ after intersecting with Φ_{n-1}. Now Proposition 6.2 applied to w' gives the equality $N(y) = N(w')$ and so $w' = y$. It follows that $w = x_i w'$, showing uniqueness of S. □

Proof of Theorem 4.1. By the previous lemma, if we consider the sum $\sum_S t^{|S|}$ over all $S \in \mathcal{W}^{\mathcal{I}}$ with $S \cap \Phi_{n-1} = S'$ for some $S' \in \mathcal{W}^{\mathcal{I}'}$, then the sum equals

$$t^{|S'|}(1 + t + t^2 + \cdots + t^{m_k}).$$

Thus

$$\begin{aligned}\sum_{S \in \mathcal{W}^{\mathcal{I}}} t^{|S|} &= \sum_{S' \in \mathcal{W}^{\mathcal{I}'}} t^{|S'|}(1 + t + t^2 + \cdots + t^{m_k}) \\ &= (1 + t + t^2 + \cdots + t^{m_k}) \prod_{i=1}^{k-1} (1 + t + t^2 + \cdots + t^{m_i}),\end{aligned}$$

where the last step is by induction. This completes the proof of the theorem in types A_n, B_n, C_n. This proof also works in type G_2. In types F_4 and E_6, the theorem was checked on a computer, running through all possible ideals. There are 105 ideals in F_4 and 833 of them in E_6.

9. A uniform proof for the penultimate ideal

The goal for this section is to prove Theorem 4.1 uniformly when $\mathcal{I} = \{\theta\}$, where θ is the highest root of Φ^+.

The next result is a special case of Theorem 2.8 in [Ma]. For γ in X (the weight lattice), let e^γ denote the corresponding element of the group algebra $\mathbb{Z}[X]$ of X.

Proposition 9.1 ([Ma])**.** *Let $R \subset \Phi^+$ be any subset of the positive roots. The following identity holds:*

$$\sum_{w \in W} \prod_{\alpha \in R} \frac{1 - te^{w\alpha}}{1 - e^{w\alpha}} = \sum_{w \in W} t^{|N(w) \cap R|}. \tag{8}$$

Proof. In Theorem 2.8 of [Ma], set $u_\alpha = t$ if $\alpha \in R$ and set $u_\alpha = 1$ if $\alpha \notin R$. □

When $R = \mathcal{I}^c$ for some ideal $\mathcal{I}$, the right-side of (8) is the Poincaré polynomial (after replacing t by t^2) of a regular semisimple Hessenberg variety (see the next section) by work of [MPS]. In that case, the identity can be proven by a fixed-point formula as in [Ma], since these Hessenberg varieties are smooth and projective (generalizing the role of the flag variety in Macdonald's fixed-point formula proof).

We now use this identity to prove Theorem 4.1 uniformly for any root system when $\mathcal{I} = \{\theta\}$.

Theorem 9.2. *In all types when $\mathcal{I} = \{\theta\}$,*

$$\sum_{S \in \mathcal{W}^{\mathcal{I}}} t^{|S|} = \prod_{i=1}^{n} (1 + t + t^2 + \cdots + t^{m_i^{\mathcal{I}}}),$$

where the $m_i^{\mathcal{I}}$ are the exponents of $\mathcal{I}$.

Proof. In (8), put $R = \mathcal{I}^c = \Phi^+ - \{\theta\}$ and specialize e^α to $t^{\,\mathrm{ht}(\alpha)}$ as in [Ma]. Then we have

$$\sum_{w\in W}\prod_{\alpha\in\mathcal{I}^c}\frac{1-t^{\,\mathrm{ht}(w\alpha)+1}}{1-t^{\,\mathrm{ht}(w\alpha)}} = \sum_{w\in W} t^{|N(w)\cap\mathcal{I}^c|}.$$

We will break apart the sum on the left side into two parts, according to whether $w \in W$ satisfies (5). Let W_{min} denote those elements of W satisfying (5) for $\mathcal{I}$.

Let $w \in W_{min}$ with $w \neq 1$. Then $w^{-1}(\alpha) \in \Phi^+$ for some $\alpha \in -\Pi$ since $w \neq 1$. Then $\beta := w^{-1}(\alpha) \in \mathcal{I}^c$ by (5) since $w \in W_{min}$. Since $\mathrm{ht}(w\beta) = -1$, the term for w vanishes in the sum on the left. This leaves only the identity term as the contribution from the elements in W_{min}.

Therefore the sum on the left side reduces to

$$\prod_{\alpha\in\mathcal{I}^c}\frac{1-t^{\,\mathrm{ht}(\alpha)+1}}{1-t^{\,\mathrm{ht}(\alpha)}} + \sum_{w\notin W_{min}}\prod_{\alpha\in\mathcal{I}^c}\frac{1-t^{\,\mathrm{ht}(w\alpha)+1}}{1-t^{\,\mathrm{ht}(w\alpha)}}. \tag{9}$$

The isolated product is exactly the right side of Theorem 9.2. To finish the proof we must, by Proposition 6.3, show that the sum in (9) is equal to

$$\sum_{w\notin W_{min}} t^{|N(w)\cap\mathcal{I}^c|}\ .$$

We divide $\mathcal{I}^c$ into two parts: let

$$\mathcal{I}^c_i = \{\gamma\in\mathcal{I}^c \mid \langle\gamma,\theta^\vee\rangle = i\}$$

for $i = 0, 1$ (the only two possibilities since θ is the highest root). The roots of $\mathcal{I}^c_0$ are the positive roots of a parabolic subsystem of Φ, with corresponding Weyl group W_θ. These are exactly the elements of W which fix θ.

Given $\gamma \in \mathcal{I}^c_1$, we have $s_\theta(-\gamma) = \theta - \gamma$, which is a positive root. Hence also $s_\theta(-\gamma) \in \mathcal{I}^c_1$ and $\gamma + s_\theta(-\gamma) = \theta$. This shows that elements of $\mathcal{I}^c_1$ come in pairs which sum up to θ.

Take $w \notin W_{min}$. Then $w\theta \in -\Pi$ and thus $\mathrm{ht}(w\theta) = -1$. Suppose that $\alpha+\beta = \theta$ for $\alpha, \beta \in \mathcal{I}^c_1$. Then

$$\mathrm{ht}(w\alpha) + \mathrm{ht}(w\beta) = -1. \tag{10}$$

Consequently, exactly one of α and β belongs to $N(w) \cap \mathcal{I}^c$, and thus

$$|N(w)\cap\mathcal{I}^c_1| = \frac{1}{2}|\mathcal{I}^c_1|$$

for all $w \notin W_{min}$.

On the other hand, the identity

$$\frac{1-t^{a+1}}{1-t^a}\cdot\frac{1-t^{-a}}{1-t^{-a-1}} = t$$

and (10) imply that

$$\prod_{\alpha\in\mathcal{I}^c_1}\frac{1-t^{\,\mathrm{ht}(w\alpha)+1}}{1-t^{\,\mathrm{ht}(w\alpha)}} = t^{\frac{1}{2}|\mathcal{I}^c_1|}$$

for $w \notin W_{min}$.

Therefore the proof will be completed if we can show that

$$\sum_{w\notin W_{min}}\prod_{\alpha\in\mathcal{I}^c_0}\frac{1-t^{\,\mathrm{ht}(w\alpha)+1}}{1-t^{\,\mathrm{ht}(w\alpha)}} = \sum_{w\notin W_{min}} t^{|N(w)\cap\mathcal{I}^c_0|}\ .$$

We can do this by using (8) for the Weyl group W_θ. First we observe that the action of W_θ on W preserves W_{min}, and so W_{min} and its complement are a union of left cosets of W_θ.

Pick $x \notin W_{min}$. Then

$$\sum_{w \in xW_\theta} \prod_{\alpha \in \mathcal{I}_0^c} \frac{1 - t^{\,\mathrm{ht}(w\alpha)+1}}{1 - t^{\,\mathrm{ht}(w\alpha)}} = \sum_{y \in W_\theta} \prod_{\alpha \in \mathcal{I}_0^c} \frac{1 - t^{\,\mathrm{ht}(xy(\alpha))+1}}{1 - t^{\,\mathrm{ht}(xy(\alpha))}}. \tag{11}$$

In (8) applied now for the case of the Weyl group W_θ, set $e^\alpha = t^{\,\mathrm{ht}(x\alpha)}$, where the height is still calculated with respect to W (since $x\alpha$ need not belong to the root system of W_θ). The positive roots for W_θ are $\mathcal{I}_0^c$, so (11) is equal to

$$\sum_{y \in W_\theta} t^{|N(y) \cap \mathcal{I}_0^c|},$$

which is the same thing as

$$\sum_{w \in xW_\theta} t^{|N(w) \cap \mathcal{I}_0^c|}$$

for any $x \in W$. Indeed, we may choose x such that $N(x) \subseteq \Phi^+ - \mathcal{I}_0^c$ by Lemma 5.3. Then Lemma 5.1 implies that $N(xw_1) \cap \mathcal{I}_0^c = N(w_1) \cap \mathcal{I}_0^c$ for $w_1 \in W_\theta$, which is what we needed.

The proof is completed since W_{min} is a union of left cosets of W_θ. □

10. Poincaré polynomials of regular nilpotent Hessenberg varieties

The combinatorial Poincaré polynomials from Theorem 4.1 should arise as the actual topological Poincaré polynomials of certain projective subvarieties of the flag variety. This section defines these subvarieties, called regular nilpotent Hessenberg varieties, and lists some of their main properties.

Write B^- for the Borel subgroup opposite to B and $\mathfrak{b}^-$ for its Lie algebra.

Given an ideal $\mathcal{I}$, we define

$$H_{\mathcal{I}} = \mathfrak{b}^- \oplus \bigoplus_{\alpha \in \mathcal{I}^c} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the α-weight space. The subspace $H_{\mathcal{I}}$ is stable for the action of B^-, and it is easy to see that each subspace with this property is of the above form. Such a subspace is called a Hessenberg space.

Fix an element $X \in \mathfrak{g}$ and a Hessenberg space $H = H_{\mathcal{I}}$. The Hessenberg variety $\mathcal{B}_{X,H}$ is the subvariety of the flag variety $\mathcal{B} = G/B^-$ defined by

$$\mathcal{B}_{X,H} = \{gB^- \mid \mathrm{Ad}(g^{-1})(X) \in H\}.$$

This is a closed subvariety of $\mathcal{B}$ and hence is projective. In general a Hessenberg variety is not smooth. Hessenberg varieties were first defined in [MPS].

When $\mathcal{I} = \Phi^+$ and thus $H = \mathfrak{b}^-$, the Hessenberg variety reduces to a Springer variety, a well-studied and important object in representation theory. At the other end of the spectrum, when $\mathcal{I} = 0$, the Hessenberg variety is the whole flag variety, independent of X. In between, when $\mathcal{I}^c$ is the set of simple roots and X is regular nilpotent, the Hessenberg variety is called the Peterson variety and has been used to give geometric constructions for the quantum cohomology of the flag variety (see [Ko3], [R]). Other Hessenberg varieties have been used in [GKM] to give a partial proof of the fundamental lemma of the Langlands program.

The following proposition concerning $\mathcal{B}_{X,H}$ follows from work in [Ty]. Let $B^-wB^- \subset \mathcal{B}$ be the Schubert cell containing the point wB^-, where $w \in W$. Here we do not distinguish between $w \in W$ and a representative of w in G.

Proposition 10.1. *Let G be of classical type and let X be a sum of negative simple root vectors. Let $C_w := B^-wB^- \cap \mathcal{B}_{X,H_\mathcal{I}}$. Then C_w is non-empty if and only if w satisfies* (5) *for $\mathcal{I}$. If C_w is non-empty, then it is an affine space of dimension $|N(w) \cap \mathcal{I}^c|$.*

This yields an affine paving of $\mathcal{B}_{X,H_\mathcal{I}}$.

This proposition allows us to demonstrate that the Poincaré polynomials for regular nilpotent Hessenberg varieties are the polynomials which arise in Theorem 4.1.

Theorem 10.2. *Let $P_\mathcal{I}(t)$ denote the Poincaré polynomial of the Hessenberg variety $\mathcal{B}_{X,H_\mathcal{I}}$ for X a regular nilpotent element. In types A_n, B_n, and C_n this can be factored*

$$P_\mathcal{I}(t^{\frac{1}{2}}) = \prod_{i=1}^{k}(1 + t + \cdots + t^{m_i^\mathcal{I}}).$$

Proof. Proposition 6.3 shows that for each $S \in \mathcal{W}^\mathcal{I}$ there is exactly one $w \in W$ satisfying (5) and $S = N(w) \cap \mathcal{I}^c$. Proposition 10.1 shows that the dimension of the affine cell C_w is $|N(w) \cap \mathcal{I}^c| = |S|$ and that C_w is empty if w does not satisfy (5). The proof follows from the fact that these cells give an affine paving of the variety together with Theorem 4.1. □

We should mention that if one knew that Proposition 10.1 were true in all types, then a result of Peterson announced in [BC] would be equivalent to Theorem 4.1. Unfortunately Peterson's proof is not given. One could imagine something along the lines of [AC] if the regular nilpotent Hessenberg varieties were smooth. But this is not the case, already in type A_2.

11. Hyperplane arrangement defined by an ideal

The second venue where the $\mathcal{I}$-exponents arise is in the context of hyperplane arrangements. Let $V := Q^\vee \otimes \mathbf{R}$ be the ambient vector space containing the coroot lattice $Q^\vee$. For each $\alpha \in \Phi^+$, let $H_\alpha \subset V$ be the hyperplane

$$H_\alpha = \{v \in V \mid \langle \alpha, v \rangle = 0\}.$$

We are interested in the hyperplane arrangement in V given by the hyperplanes H_α, where $\alpha \in \mathcal{I}^c$. We will denote this arrangement by $\mathcal{A}_\mathcal{I}$ and call it an arrangement of ideal type in Φ_n.

In general, given a hyperplane arrangement, one is interested in whether the arrangement is free, and if so, what the roots are of its characteristic polynomial, which are also called exponents [OT].

We briefly recall the basic definitions and theorems about hyperplane arrangements from Chapters 2 and 4 of [OT]. Let $\mathcal{A}$ be an arrangement of hyperplanes in the $\mathbb{R}$-vector space V. Let $S(V^*)$ denote the symmetric algebra on the dual space V^* of V. Given $H \in \mathcal{A}$, let $\alpha_H \in V^*$ be a non-zero linear functional vanishing on H. Set

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H.$$

Let $D(\mathcal{A})$ denote the $\mathbb{R}$-linear derivations of $S(V^*)$ which preserve the ideal generated by $Q(\mathcal{A})$. Then the hyperplane arrangement $\mathcal{A}$ is said to be free if $D(\mathcal{A})$ is a free $S(V^*)$-module.

Let $L = L(\mathcal{A})$ denote the set of non-empty intersections of the elements of $\mathcal{A}$. This is a poset with a partial order given by reverse inclusion, with minimal element V. Define a function μ on L as follows. Set $\mu(V) = 1$ and define $\mu(X)$ recursively for $X \in L$ by the formula

$$\mu(X) = -\sum_{V \leq Z < X} \mu(Z),$$

where the sum is over all $Z \in L$ with $V \leq Z < X$. Then the characteristic polynomial $\chi(\mathcal{A}, t)$ of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) t^{\dim(X)}.$$

The factorization result of Terao (see Theorem 4.137 in [OT]) states that if $\mathcal{A}$ is free, then all of the roots of $\chi(\mathcal{A}, t)$ are non-negative integers, called the exponents of $\mathcal{A}$. These exponents coincide with the polynomial degrees of a set of homogeneous generators of $D(\mathcal{A})$ as an $S(V^*)$-module.

There is another key property of hyperplane arrangements. Given $H_0 \in \mathcal{A}$, let $\mathcal{A}'$ denote the arrangement in V obtained by omitting the hyperplane H_0 from $\mathcal{A}$. This is the deleted arrangement given by H_0. Let $\mathcal{A}''$ denote the arrangement in H_0 given by the non-empty intersections $H \cap H_0$ for $H \in \mathcal{A}$ with $H \neq H_0$. This is the restricted arrangement given by H_0. The three arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ are called a triple of arrangements.

We will use the following direction of the Addition-Deletion Theorem of Terao in what follows (see Theorem 4.51 in [OT]). If $\mathcal{A}'$ is free with exponents $b_1, \ldots, b_{k-1}$, $b_k - 1$ and if $\mathcal{A}''$ is free with exponents $b_1, \ldots, b_{k-1}$, then $\mathcal{A}$ is free with exponents $b_1, \ldots, b_k$. If we wished only to know about the implication involving the exponents, this result goes back to Brylawski and Zaslavsky (see Theorem 2.56 in [OT]).

Theorem 11.1. *Except possibly in types F_4, E_6, E_7, E_8, the hyperplane arrangement $\mathcal{A}_{\mathcal{I}}$ is free, and its non-zero exponents are $m_1^{\mathcal{I}}, \ldots, m_k^{\mathcal{I}}$. There are also $n-k$ exponents equal to* 0.

Proof. We will show that $\mathcal{A}_{\mathcal{I}}$ is free with the desired exponents by using the Addition-Deletion Theorem. First assume that X_n is of type A_n, B_n, C_n and assume the result for any ideal $\mathcal{I}_1$ properly containing $\mathcal{I}$. Furthermore assume the result for root systems of smaller rank of these types. The theorem is clearly true for the base case where $\mathcal{I} = \Phi^+$, since the arrangement is empty.

Let $\mathcal{I}_1$ be the unique ideal for which $\mathcal{I}_1 = \mathcal{I} \cup \{\delta\}$, where δ is the maximal root in $\Phi^1 \cap \mathcal{I}^c$. Of course, if the latter intersection is empty, then we are already done, since the arrangement is the direct product of the one-dimensional empty arrangement and an arrangement of ideal type in Φ_{n-1}.

Now by induction $\mathcal{A}_{\mathcal{I}_1}$ is free and its non-zero exponents are

$$m_1^{\mathcal{I}}, \ldots, m_k^{\mathcal{I}} - 1$$

(we do not order the exponents), where we have assigned $m_k^{\mathcal{I}} = \operatorname{ht}(\delta)$.

Next consider the restricted arrangement defined by H_δ. This is the arrangement in H_δ defined by the hyperplanes $H_\alpha \cap H_\delta$ for $\alpha \in \mathcal{I}^c$ and $\alpha \neq \delta$. We denote this restricted arrangement by $\mathcal{A}^\delta$. For $\beta \in \Phi^1$ and $\beta \prec \delta$ (6) says that $\delta - b\beta = c\gamma$, where $\gamma \in \Phi_{n-1}$. It follows that either γ or $-\gamma$ is in $\mathcal{I}^c$ since $\delta \in \mathcal{I}^c$. Set $\mathcal{I}' = \mathcal{I} \cap \Phi_{n-1}$. Then the hyperplane arrangement $\mathcal{A}_{\mathcal{I}'}$ (defined for Φ_{n-1}) is isomorphic to $\mathcal{A}^\delta$. Indeed, for each $\beta \in \Phi^1$ with $\beta \neq \delta$ we have

$$H_\beta \cap H_\delta = H_\gamma \cap H_\delta$$

for some $\gamma \in \mathcal{I}'^c$. Thus the hyperplanes $H_\gamma \cap H_\delta$ for $\gamma \in \mathcal{I}'^c$ yield the distinct hyperplanes in $\mathcal{A}^\delta$.

Thus $\mathcal{A}^\delta$ is free and its non-zero exponents are

$$m_1^{\mathcal{I}}, m_2^{\mathcal{I}}, \dots, m_{k-1}^{\mathcal{I}},$$

since these are the exponents of $\mathcal{I}'$.

Consequently by the Addition-Deletion Theorem applied to the triple of arrangements $(\mathcal{A}_{\mathcal{I}}, \mathcal{A}_{\mathcal{I}_1}, \mathcal{A}^\delta)$, the arrangement $\mathcal{A}_{\mathcal{I}}$ is free and its non-zero exponents are equal to

$$m_1^{\mathcal{I}}, m_2^{\mathcal{I}}, \dots, m_k^{\mathcal{I}},$$

as desired.

Type G_2 is trivial. We consider the case of type D_n. Here,

$$\Phi^1 = \{e_1 \pm e_j \mid 2 \leq j \leq n\},$$

using the standard notation for roots in D_n. Let

$$\begin{aligned} \gamma_1 &= \quad e_1 + e_n, \\ \gamma_2 &= \quad e_1 - e_n. \end{aligned}$$

The above proof carries over perfectly well as long as γ_1 and γ_2 do not both belong to $\mathcal{I}^c \cap \Phi^1$, since (6) would hold for $\alpha, \beta \in \mathcal{I}^c \cap \Phi^1$.

Suppose that $\gamma_1, \gamma_2 \in \mathcal{I}^c$. First, assume that both γ_1 and γ_2 are maximal elements of $\mathcal{I}^c \cap \Phi^1$. Then $\mathcal{I}^c$ consists of the n non-empty sets:

(12)

$$\begin{aligned} &\{e_{1+i} - e_j \mid 2+i \leq j \leq n\} \cup \{e_{1+i} + e_j \mid a_i \leq j \leq n-1\} \text{ with } 1 \leq i \leq n-2, \\ &\{e_1 - e_j \mid 2 \leq j \leq n\}, \\ &\{e_j + e_n \mid 1 \leq j \leq n-1\}, \end{aligned}$$

where a_i is some natural number satisfying $2 + i \leq a_i \leq n-1$. The former sets contain one root of each height $1, \dots, m_i^{\mathcal{I}}$, where $m_i^{\mathcal{I}}$ depends on a_i. The latter two sets contain one root of each height $1, \dots, n-1$, and so the ideal exponents of $\mathcal{I}$ can be written as

$$m_1^{\mathcal{I}}, m_2^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, n-1.$$

Let $\mathcal{I}_1 = \mathcal{I} \cup \{\gamma_1\}$. By induction $\mathcal{A}_{\mathcal{I}_1}$ is free and its non-zero exponents are

$$m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, n-2.$$

Consider the restricted arrangement $\mathcal{A}^{\gamma_1}$ defined by H_{γ_1}. We want to show it is free with non-zero exponents $m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1$.

In order to do this, consider the deleted and restricted arrangements of $\mathcal{A}^{\gamma_1}$ defined by H_{γ_2}. The deleted arrangement $(A^{\gamma_1})'$ is isomorphic to the arrangement defined by $\mathcal{I}' = \mathcal{I} \cap \Phi_{n-1}$ in Φ_{n-1} by the same proof as in the other classical cases.

Thus $(A^{\gamma_1})'$ is free and its non-zero exponents

$$m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-2$$

by inspection of the heights of roots in (12).

On the other hand, the restricted arrangement $(A^{\gamma_1})''$ defined by H_{γ_2} is more complicated. This arrangement lives in $H_{\gamma_1} \cap H_{\gamma_2}$, which coincides with the intersection of the null spaces of e_1 and e_n. The hyperplanes defining $(A^{\gamma_1})''$ are given by the null spaces of

$$\{e_{1+i} - e_j \mid 2+i \leq j < n\} \cup \{e_{1+i}\} \cup \{e_{1+i} + e_j \mid a_i \leq j \leq n-1\},$$

for $i = 1, \dots, n-2$. This arrangement is precisely an ideal arrangement in B_{n-2} which is free with non-zero exponents equal to

$$m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}.$$

By the Addition-Deletion Theorem, $\mathcal{A}^{\gamma_1}$ is free with non-zero exponents equal to

$$m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1,$$

and using the theorem a second time, it follows that $\mathcal{A}_{\mathcal{I}}$ is free and its non-zero exponents are

$$m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, n-1,$$

as desired.

Finally, consider the case where

$$\delta = e_1 + e_{2n-1-k}$$

is the maximal element of $\mathcal{I}^c \cap \Phi^1$ for $k > n-1$. In this case $\mathcal{A}_{\mathcal{I}_1}$ is free with non-zero exponents

$$m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, k-1,$$

by induction.

It suffices to complete the proof by showing that the restricted arrangement of $\mathcal{A}_{\mathcal{I}}$ defined by H_δ is isomorphic to $\mathcal{A}^{\gamma_1}$ above. Consider the element w of the Weyl group of D_n given by exchanging e_n and e_{2n-1-k} and fixing all other e_i. It is not hard to check that the hyperplanes defining $\mathcal{A}^{\gamma_1}$ in H_{γ_1} are mapped to the hyperplanes defining this restricted arrangement in H_δ, yielding the isomorphism and completing the proof in type D_n. □

We note that a uniform proof in all types for the case $\mathcal{I} = \{\theta\}$ is easy. On the one hand, the full Coxeter arrangment is free with exponents the usual exponents $m_1 \leq \cdots \leq m_n$. On the other hand, the restricted arrangement for H_θ is free with exponents $m_1, \dots, m_{n-1}$ by [OST]. Thus $\mathcal{A}_{\mathcal{I}}$ is free with the desired exponents.

Remark 11.2. The question of which subarrangements of a Coxeter arrangement are free and which are not has been addressed in [ER] for type A_n and certain subarrangements in type B_n. In general, it is not true that such a subarrangement is free.

12. Speculation

The two main theorems of this paper are likely to be equivalent by a general principle. Namely, suppose a hyperplane arrangement $\mathcal{A}$ is free with exponents $m_1, \ldots, m_n$. Suppose further that the arrangement is central, meaning each hyperplane contains the origin.

Let $\mathcal{C}_{\mathcal{A}}$ denote the set of components of the complement $V - \bigcup_{H \in \mathcal{A}} H$. Fix one component $A \in \mathcal{C}_{\mathcal{A}}$. Then for each component $B \in \mathcal{C}_{\mathcal{A}}$ we can define $l(B)$ to be the least number of hyperplanes needed to be crossed to move from B to A.

Define a polynomial

$$P_{\mathcal{A}}(t) = \sum_{B \in \mathcal{C}_{\mathcal{A}}} t^{l(B)}.$$

This is equivalent to the left-hand side of (1) in the case when $\mathcal{A}$ is of ideal type and the component A is chosen to contain the dominant Weyl chamber.

One might wonder what conditions on $\mathcal{A}$ would ensure that there exists a component A such that

$$P_{\mathcal{A}}(t) = \prod_{i=1}^{n} (1 + t + t^2 + \cdots + t^{m_i}).$$

Supersolvable arrangements are known to possess this property [BEZ] (see also Stanley's Park City Notes [St]). Our class of ideal hyperplane arrangements are not in general supersolvable.

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